MATRICES & DETERMINANTS

EXERCISE – I

HINTS & SOLUTIONS

Sol.1 C

The matrix will be any one of the following type
$1 \times 12,\ 12 \times 1,\ 2 \times 6,\ 6 \times 2,\ 3 \times 4,\ 4 \times 3$

Sol.2 A

Given, $\begin{bmatrix} x^2+x & x \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$

$\therefore\ x^2 + x = 0,\ x - 1 = -2,\ -x + 4 = 5,\ x + 2 = 1$
$\Rightarrow\ x = -1$

Sol.3 D

Given $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$

order of A is 3 × 1 and B is 3 × 3 product of AB is defined only when the number of columns in A is equal to the number of rows in B so AB does not exist.

Sol.4 A

Given $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

then $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos\theta + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sin\theta$

$= I\cos\theta + J\sin\theta$

Sol.5 B

Given A and B are square matrices of order 2
then $(A + B)^2 = (A + B)(A + B)$
$= A.A + A.B + B.A + B.B$
$= A^2 + AB + BA + B^2$

Sol.6 C

Since in a skew symmetric matrix all elements along the principal diagonal are zero. If A is skew symmetric matrix then trace of A is zero.

Sol.7 A

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then adj $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

Sol.8 A

Given $A = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

then adj $A = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = A$

Sol.9 B

Given A is a square matrix such that $A^2 = I$
$\Rightarrow\ AA = I$
By definition of inverse $AA^{-1} = A^{-1}A = I$ then $A^{-1} = A$

Sol.10 B

Given $|A| = -1$ & $|B| = 3$
$|3\,AB| = 3^3(-1)\,3$
$= -81$

Sol.11 C

Given $A = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

Now $|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) - 2(-3+4) + 0(-12+12) = 1$

adj $A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

then $A^{-1} = \dfrac{\text{adj}\,A}{|A|} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$

and $A^3 = A.A.A.$

$= \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

Hence $A^{-1} = A^3$

Sol.12 A

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and A = Bx

Let $x = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

then $A = Bx \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ 2b & 2d \end{bmatrix}$

$\therefore$ $a = 1, b = \frac{3}{2}, c = 2, d = -\frac{5}{2}$

Hence $x = \begin{bmatrix} 1 & 2 \\ 3/2 & -5/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$

Sol.13 C

$\det(B^{-1}AB) = \det(B^{-1}) \det(A) \det(B)$
$= (\det(B))^{-1} \det(A) \det(B)$
$= \det(A)$

Sol.14 A

Given equations are $-2x + y + z = 1$
$x - 2y + z = -2$
$x + y + \lambda z = 4$

Here $D = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix}$

$D = -2(-2\lambda - 1) - 1(\lambda - 1) + 1(1 + 2)$
$D = 3\lambda + 6$

The system of equation will have no solution if D = 0
so $3\lambda + 6 = 0 \Rightarrow \lambda = -2$

Sol.15 B

Given equations are $x + y - z = 6$
$x + 2y - 3z = 14$
$2x + 5y - \lambda z = 9$

Here $D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ 2 & 5 & -\lambda \end{vmatrix}$

$D = 1(-2\lambda + 15) - 1(-\lambda + 6) - 1(5 - 4)$
$D = -\lambda + 8$

The system of equation has a unique solution if $D \neq 0$ so $-\lambda + 8 \neq 0 \Rightarrow \lambda \neq 8$

Sol.16 D

Given equations are $x + 2y + 3z = 4$
$x + \lambda y + 2z = 3$
$x + 4y + \mu z = 3$

Here $D = \begin{vmatrix} 1 & 2 & 3 \\ 1 & \lambda & 2 \\ 1 & 4 & \lambda \end{vmatrix}$

$D = (\lambda\mu - 8) - 2(\mu - 2) + 3(4 - \lambda)$
$= \lambda\mu - 2\mu - 3\lambda + 8$
$= (\lambda - 2)\mu - 3\lambda + 8$

The system of equation has an infinite a number of solution if D = 0 which not gives value of λ and μ.

Sol.17 C

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & 3 & -3 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & 3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 8 & 24 \end{bmatrix}$

$A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 8 & 24 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 12 & 36 \end{bmatrix}$

$A^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 12 & 36 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

Hence, A is nilpotent matrix of order 5.

Sol.18 B

Given A = diag(2, –1, 3), B = diag(–1, 3, 2)
then A^2B = A.A.B.
= diag (2, –1, 3) diag (2, –1, 3) diag(–1, 3, 2)
= diag (4, 1, 9) diag(–1, 3, 2)
= diag (–4, 3, 18)

Sol.19 B

Given $A = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$

then $A^T = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$ and $B^T = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$

Now, $B^TA^T = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

which an identity matrix.

Sol.20 C

If the matrix AB is a zero matrix then
It is not necessary that either A = 0 or B = 0

Sol.21 D

Given $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$

then $AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & -2 \end{bmatrix}$

and $BA = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -3 & 3 \end{bmatrix}$

so $AB \neq BA$

Hence $(A + B)^2 \neq A^2 + 2AB + B^2$

$(A - B)^2 \neq A^2 - 2AB + B^2$

Sol.22 A

Given AB = A & BA = B

$AB = A$ ....(1)

$BA = B$ ....(2)

(1) × (2)

$AB^2A = AB$

$B^2A = B$

$B^2 = BA^{-1}$ ....(3)

Consider (2)

$BA = B$

$B = BA^{-1}$ ....(4)

SDo, from (3)

$B^2 = B$

Sol.23 A

$(ABA)^T = A^TB^TA^T$

= ABA ($\because$ A & B are symmetric matrices)

So, ABA is symmetric matrix.

Sol.24 A

Let $A^n = P$

Taking transpose both sides.

$(A^n)^T = P^T$

$(A^T)^n = P^T \Rightarrow P^T = (-A)^n = A^n$

$\Rightarrow P^T = P$ Hence symmetric matrix.

Sol.25 D

Given A is a non-singular matrix, i.e. $|A| \neq 0$

So, $|A^T| \neq 0$ So, $|A| + |A^T| \neq 0$

Sol.26 C

$(AB)^n = A^nB^n$

Given that AB = BA

Far n = 1 :

$(AB)^1 = A^1B^1$

For n = m

$(AB)^m = A^m B^m$

For n = m + 1 :

$(AB)^{m+1} = A^{m+1} B^{m+1}$

LHS

$(AB)^{m+1} = (AB)^m (AB)$

$= (A^mB^m)(AB)$

$= A^m(B^m(AB))$

$= A^m (B^m(BA))$ ($\because$ AB = BA)

$= A^m (B^{m+1}A)$

$= A^m(AB^{m+1}) = A^{m+1} B^{m+1}$

Sol.27 A

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$-A = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$

det (A) = – det A

Sol.28 A

$(\alpha + a)x + \alpha y + \alpha z = 0$

$\alpha x + (\alpha + b)y + \alpha z = 0$

$\alpha x + \alpha y 1 - (\alpha + c) z = 0$

From non-trivial solution,

D = 0

$$\begin{bmatrix} \alpha + a & \alpha & \alpha \\ \alpha & \alpha + b & \alpha \\ \alpha & \alpha & \alpha + c \end{bmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 3\alpha + \alpha & 3\alpha + b & 3\alpha + c \\ \alpha & \alpha + b & \alpha \\ \alpha & \alpha & \alpha + c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3\alpha & 3\alpha & 3\alpha \\ \alpha & \alpha + b & \alpha \\ \alpha & \alpha & \alpha + c \end{vmatrix} + \begin{vmatrix} a & b & c \\ \alpha & \alpha + b & \alpha \\ \alpha & \alpha & \alpha + c \end{vmatrix} = 0$$

$$\Rightarrow 3a \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \alpha + b & \alpha \\ \alpha & \alpha & \alpha + c \end{vmatrix} + \begin{vmatrix} a & b & c \\ \alpha & \alpha + b & \alpha \\ \alpha & \alpha & \alpha + c \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ & $R_2 \to R_2 - R_3$

Both parts respectively

$$\Rightarrow 3a \begin{vmatrix} 1 & 0 & 0 \\ \alpha & b & 0 \\ \alpha & 0 & c \end{vmatrix} + \begin{vmatrix} a & b & c \\ 0 & b & c \\ \alpha & \alpha & \alpha + c \end{vmatrix} = 0$$

$\Rightarrow \alpha(ab + bc + ca) + ab\,c = 0$

$\Rightarrow \alpha(-c^{-1} - a^{-1} - b^{-1}) = +1$

$\Rightarrow \alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$

Sol.29 B

Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - I \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}$

$|P| = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0 = (1-\lambda)(2-\lambda) - 6 = 0$

$= 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0 \quad = \lambda^2 - 3\lambda - 4 = 0$

Sol.30 B

AB = AC

pre-multiply by A^{-1} $\Rightarrow$ B = C

for existance of A^{-1} $\Rightarrow$ $|A| \neq 0$

$\Rightarrow$ A is non-singular matrix

Sol.31 D

$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$

$B = A^{-1} \Rightarrow AB = I_3$

$10BA = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$10I_3 = \begin{bmatrix} 4+4+2 & -4+2+2 & 4-6+2 \\ -5+\alpha & 5+\alpha & -5+\alpha \\ 1-4+3 & -1-2+3 & 1+6+3 \end{bmatrix}$

$19I_3 = \begin{bmatrix} 10 & 0 & 0 \\ \alpha-5 & \alpha+5 & \alpha-5 \\ 0 & 0 & 10 \end{bmatrix}$

$10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ \alpha-5 & \alpha+5 & \alpha-5 \\ 0 & 0 & 10 \end{bmatrix}$

$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ \alpha-5 & \alpha+5 & \alpha-5 \\ 0 & 0 & 10 \end{bmatrix} \Rightarrow \alpha = 5$

Sol.32 A

3x +ky – 2z = 0

x + ky + 3z = 0

2x + 3y – 4z = 0

D = 0

$\begin{vmatrix} 3 & k & -2 \\ 1 & k & 3 \\ 2 & 3 & -4 \end{vmatrix} = 0$

3(–4k – a) – k (– 4 – 6) – 2(3 – 2k) = 0

–12k – 27 + 4k + 6k – 6 + 43k = 0

2k = 33

$k = \frac{33}{2}$

Sol.33 A

$a^3x + (a+1)^3y + (a+2)^3z = 0$

$ax + (a+1)y + (a+2)z = 0$

$x + y + z = 0$

D = 0

$\begin{vmatrix} a^3 & (a+1)^3 & (a+2)^3 \\ a & (a+1) & (a+2) \\ 1 & 1 & 1 \end{vmatrix} = 0$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$\begin{vmatrix} a^3 & (a+1)^3 - a^3 & (a+2)^3 - a^3 \\ a & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$

$\{2(a+1)^3 - 2a^3 - (a+2)^3 + a^3\} = 0$

$\{(a+1)^3 - (a+2)^3 - a^3\} = 0$

$a^3 + 1 + 3a + 3a^2 - a^3 - 8 - 8 - 6a^2 - 12a - a^3 = 0$

$-a^3 - 3a - 9a - 7 = 0$

$a^3 + 3a^2 + aa + 7 = 0$

a = –1

Sol.34 D

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let,

$f(x) = x^2 - (a+d)x + k = 0$

$f(A) = A^2 - (a+d)A + k = 0$ ....(1)

Becuase 'A' satisfies f(x)

$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ab+d^2 \end{bmatrix}$

$(a+d)A = (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ab+d^2 \end{bmatrix}$

$k = kI_2 = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Putting in (1)

$\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ab+d^2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = 0$

$k = a^2 + ad - a^2 - bc$

k = ad – bc

Sol.35 C

Let $M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$M^2 = M.M. = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$M^2 = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = 0$

Sol.36 A

$2x + y = 4$
$3x + 2y = 2$
$x + y = 2$

$$D = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

$$D_1 = \begin{vmatrix} 4 & 1 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 2 & 4 & 0 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} \neq 0$$

Hence no solution

Sol.37 D

Given A is a square matrix,

$$\text{So let } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$A' = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$P = A + A' = \begin{bmatrix} 2a & b+d & c+g \\ d+b & 2e & f+h \\ g+c & f+h & ei \end{bmatrix}$$

$P^T = P \Rightarrow$ symmetric

Similarly $A'A$ & AA' also come out to be symmetric

$$\text{Now } Q = A - A' = \begin{bmatrix} 0 & b-d & c-g \\ d-b & a & f-h \\ g-c & h-f & 0 \end{bmatrix}$$

$Q^T = -Q \Rightarrow$ Skew symmetric

Sol.38 B

$$[1 \times 1] \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = [0]$$

$$[1 + 0 + 0 \quad 3 + 5x + 3 \quad 2 + x + 2] \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = [0]$$

$$[1 + 5x + 6 \quad x + 4] \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = [0]$$

$$[x + 5x + 6 \quad x + 4] \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = [0]$$

$[x + 5x + 6 - 2x - 8] = 0$

$4x - 2 = 0 \Rightarrow x = \frac{1}{2}$

Sol.39 C

Given $AB = B$, $BA = A$

$AB = B$

Premuttiply with 'B',

$B(AB) = B^2 \Rightarrow (BA)\,B = B^2 \Rightarrow AB = B^2$ ....(1)

Now, $BA = A$

Premultiply with 'A',

$A(BA) = A^2 \Rightarrow (AB)\,A = A^2 \Rightarrow BA = A^2$ ....(2)

(1) + (2)

$AB + BA = B^2 + A^2$

$B + A = B^2 + A^2$

Sol.40 D

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

Given $(A + B)^2 = A^2 + B^2 + 2AB$

$$\begin{bmatrix} 1+a & 0 \\ 2+B & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^2 + \begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}^2 + 2\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$\begin{bmatrix} (1+a)^2+0 & 0+0 \\ (2+b)(1+a) & 0+0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$+ 2\begin{bmatrix} a-b & 1+1 \\ 2a+b & 2-1 \end{bmatrix}$$

$$\begin{bmatrix} a^2+1 & 0 \\ ab+2a+b+2 & 0 \end{bmatrix} = \begin{bmatrix} a^2+b+3+2a-2b & -2+a-1+4 \\ 4+ab-b+4a+2b & -1+b+1+2 \end{bmatrix}$$

$a = -1$
$b = -2$

Sol.41 A

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A' = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 9+1+1 & 0+1-2 \\ 0+1-2 & 0+1+4 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix}$$

Let $AA' = P$

$$P^T = \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix} = P \qquad \text{Hence symmetric}$$

Sol.42 A

$x + y + z = 8$
$x - y + 2z = 6$
$3x + 5y - 7z = k_1$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{vmatrix} = 1(7-10) - (-7-6) + 1(5+3)$$

$= -3 + 13 + 15 = 25$

$$D_1 = \begin{vmatrix} 8 & 1 & 1 \\ 6 & -1 & 2 \\ 19 & 5 & -7 \end{vmatrix} = 8(7-10) - 1(-42-28) + (30+14)$$

$= -24 + 70 + 44 = 90$

$$D_2 = \begin{vmatrix} 1 & 8 & 1 \\ 1 & 6 & 2 \\ 3 & 5 & 14 \end{vmatrix} = 1(-14-30) - (14+8) + 2(5+3)$$

$= -44 + 4 + 64 = 24$

$x = \frac{90}{25}, y = \frac{90}{25}, z = \frac{24}{25}$ Hence unique solution

Sol.43 B

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

$$C_{11} = \begin{vmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{vmatrix} = \omega^2 - \omega^4 = (\omega^2 - \omega)$$

$$C_{12} = (-1)\begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix} = \omega^2 - \omega$$

$$C_{13} = \begin{vmatrix} 1 & \omega \\ 1 & \omega^2 \end{vmatrix} = \omega^2 - \omega$$

$$C_{21} = (-1)\begin{vmatrix} 1 & 1 \\ \omega^2 & \omega \end{vmatrix} = \omega^2 - \omega$$

$$C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1$$

$$C_{23} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = 1 - \omega^2$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ \omega & \omega^2 \end{vmatrix} = \omega^2 - \omega$$

$$C_{32} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = (\omega^2 - 1) = 1 - \omega^2$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1$$

$$C = \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & -(\omega^2-1) & \omega-1 \end{bmatrix}$$

$$\text{adj } A = c^T = \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & -(\omega^2-1) \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \Rightarrow 1\{\omega-\omega\} - 1\{\omega-\omega\} + 1\{\omega^2-\omega\}$$

$\Rightarrow \omega^2 - \omega - \omega + \omega^2 + \omega^2 - \omega$
$\Rightarrow 3(\omega^2 - \omega) \Rightarrow 3\omega(\omega - 1)$

$$A^{-1} = \frac{\text{adj. } A}{|A|}$$

$$= \frac{1}{3\cos(\omega-1)}\begin{bmatrix} \omega(\omega-1) & \omega(\omega-1) & \omega(\omega-1) \\ \omega(\omega-1) & \omega-1 & -(\omega-1)\omega+1 \\ \omega(\omega-1) & (1-\omega)(1+\omega) & \omega-1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{\omega} & -(\omega+1) \\ 1 & -(1+\omega) & \frac{1}{\omega} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & +\omega^2 \\ 1 & +\omega^2 & \omega^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

Sol.44 C

$$A = \begin{bmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{bmatrix}$$

$$|A| = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 3x+\lambda & 3x+\lambda & 3x+\lambda \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

$$= (3x+\lambda) \begin{vmatrix} 1 & 1 & 1 \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$= (3x+\lambda) \begin{vmatrix} 1 & 0 & 0 \\ x & \lambda & 0 \\ x & 0 & \lambda \end{vmatrix}$$

$= (3x + \lambda) [1 (\lambda^2 - 0) + 0 + 0]$

New $|A| \neq 0$ for existence of A^{-1}

So, $3x + \lambda \neq 0$ & $\lambda \neq 0$

Sol.45 C

$$A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

Given $0 \in [0, 2\pi]$

$$\text{Det } A = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

$= 1(1 + \sin^2\theta) - \sin q\,(-\sin\theta + \sin\theta) + 1\,(\sin^2\theta + 1)$

$= 1 + \sin^2\theta + \sin^2\theta + 1$

$= 2\sin^2\theta + 2$

$= 2(\sin^2\theta + 1)$ → at $\sin\theta = 0, \text{Det}(A) = 2$; at $\sin\theta = 0, \text{Det}(A) = 4$

Det (A) $\in [2, 4]$

Sol.46 A

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & c \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

$C_{11} = \begin{vmatrix} 2 & 3 \\ a & 1 \end{vmatrix} = 2 - 3a$

$C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$

$C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & a \end{vmatrix} = a - 6$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 2a - 1$

$C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$

$C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

$C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$

$C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$$C = \begin{bmatrix} 2-3a & 8 & a-6 \\ 2a-1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\text{adj } A = C^T = \begin{bmatrix} 2-3a & 2a-1 & -1 \\ 8 & -6 & 2 \\ a-6 & 3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

So, $|A| = \left[0 - 1\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 3 & a \end{vmatrix}\right]$

$= 0 + 8 + 2a - 12$

$= 2(a - 2)$

$$\text{Now, } A^{-1} = \frac{1}{2(a-2)} \begin{bmatrix} 2-3a & 2a-1 & -1 \\ 8 & -6 & 2 \\ a-6 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2-3a}{2a-4} & \frac{2a-1}{2a-4} & \frac{-1}{2a-4} \\ \frac{8}{2a-4} & \frac{-3}{a-2} & \frac{1}{a-2} \\ \frac{a-6}{2(a-2)} & \frac{3}{2(a-2)} & \frac{-1}{2(a-2)} \end{bmatrix} \quad ....(1)$$

But given that,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & c \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \quad ....(2)$$

So, from (1) & (2)

$$\frac{2-3a}{2(a-2)} = \frac{1}{2}$$

$2 - 3a = a - 2$

$4a = 4$

$a = 1$ & $c = \frac{1}{a-2} = \frac{1}{1-2} = -1$

$c = -1$

Sol.47 B

Given that $B = -A^{-1} BA$

Let $P = (A + B)^2 = (A + B)(A + B)$

$= A^2 + BA + AB + B^2$ ....(1)

We 've to find the value of 'P',

Consider, $B = -A^{-1}(BA)$

premultiply both sides by 'A',

$AB = -BA$ ....(2)

From (1)

$P = A^2 + B^2$

Sol.48 C

$$A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 10\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 5\alpha^2 + 25\alpha \\ 0 & 0 & 25 \end{bmatrix}$$

$|A^2| = 25(25\alpha^2) = 25$

$= \alpha^2 = \pm \dfrac{1}{5}$

$|\alpha| = \dfrac{1}{5}$

Sol.49 A

Given $A^2 - B^2 = (A - B)(A + B)$

Consider $(A - B)(A + B)$

$A^2 + AB - BA - B^2 = A^2 - B^2$

provided $AB = BA$

Sol.50 A

Given $A^2 - A + I = 0$

$A^2 = A - I$

$A^2A-1 = AA^{-1} - IA^{-1}$

$A = I - A^{-1} \quad \Rightarrow \quad A^{-1} = (I - A)$

Sol.51 D

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Put n = 2, only [D] will satisfy

Sol.52 D

$\alpha z + y + z = \alpha - 1$

$x + \alpha y + z = \alpha - 1$

$x + y + \alpha z = \alpha - 1$

$D = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \alpha(\alpha^2 - 1) - 1(-1 + \alpha) + (1 - \alpha)$

$= \alpha^3 - \alpha + 1 - \alpha + 1 - \alpha$

$= \alpha^2(\alpha + \alpha - 2)(\alpha - 1) = 0$

$= (\alpha^2 + 2\alpha - \alpha - 2)(\alpha - 1) = 0$

$= (\alpha + 2)(\alpha - 1)^2 = 0$

$= \alpha = -2$ or 1

$D_1 = \begin{vmatrix} \alpha-1 & 1 & 1 \\ \alpha-1 & \alpha & 1 \\ \alpha-1 & 1 & \alpha \end{vmatrix}$

$D_2 = \begin{vmatrix} \alpha & \alpha-1 & 1 \\ 1 & \alpha-1 & 1 \\ 1 & \alpha-1 & \alpha \end{vmatrix}, D_3 = \begin{vmatrix} \alpha & 1 & \alpha-1 \\ 1 & \alpha & \alpha-1 \\ 1 & 1 & \alpha-1 \end{vmatrix}$

If $\alpha = 1$ then, $D_1 = D_2 = D_3 = 0$

$\alpha = 2$

Sol.53 D

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$|A| = 0 + 0 - 1(-1) = 1$

$$A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Sol.54 B

$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

$\alpha = a^2 + b^2$

$\beta = 2ab$

Sol.55 C

$A = \begin{bmatrix} a & 1 \\ 0 & 0 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(aI + bA)^2 = a^2 + b^2A^2 + 2aI.bA$
$+ bA.aI$

$aI = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, bA = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

$$aI \,.\, bA = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}$$

$$bA \,.\, aI = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & ab \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}$$

So, $aI \,.\, bA = bA \,.aI = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix} = ab\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = abA$

So, $(aI + bA)^2 = a^2I^2 + b^2A^2 + 2abA$
$= a^2I + 2abA$

$(\because \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$

Sol.56 A

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence A is nilpotent.

Sol.57 A

Given A is a singular matrix, i.e. $|A| = 0$

$A(\text{adj } A) = |A|\, I_n = 0$

$(\because \; |A| = 0)$

So, A(adj. A) is a null matrix

Sol.58 C

AB = 0

Det (AB) = 0

Det (A) . Det (B) = 0

Det (A) = 0 or Det (B) = 0

Sol.59 A

Direct by property

Sol.60 C

$A^2 = I$

$A^2 A^{-1} = IA^{-1}$

$A = A^{-1} \Rightarrow$ A is an involutory matrix

Sol.61 B

Because for unique solution $D \neq 0$

where D = coefficient matrix

Sol.62 B

Given, $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \sin\theta\sin\phi\cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$

Applying $R_1 \to R_1 \cos\theta - R_2 \sin\theta$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$\Delta = 1 (\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi)$

$\Delta = \sin\theta\cos\theta(\sin^2\phi + \cos^2\phi)$

$\Delta = \sin\theta\cos\theta$

which is independent of ϕ

Sol.63 A

Let $k = \begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$

$P_2 \to R_2 - R_1 \quad \& \; R_3 \to R_3 - R_1$

$$= K = \begin{vmatrix} -1 & 2 & 1 \\ 4 = 2\sqrt{2} & 2\sqrt{2} & 0 \\ 4-2\sqrt{2} & -2\sqrt{2} & 0 \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= k = \begin{vmatrix} -1 & 2 & 1 \\ 8 & 0 & 0 \\ 4-2\sqrt{2} & -2\sqrt{2} & 0 \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$= k = -3(0) - 2(0) + 1(-16\sqrt{2})$

$= k = -16\sqrt{2}$

Absolute value will be $|k| = 16\sqrt{2}$

Sol.64 D

Given α, β, γ are the roots of

$x^3 + px + q = 0$

so, $\alpha + \beta + \gamma = 0 \quad \& \quad \alpha\beta\gamma = -q$

Let $\quad \ell = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$\ell = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix}$$

$$\ell = \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

Sol.65 D

Given a, b, c > 0 & x, y, z $\in$ R, then

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$= \begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^y-b^{-y})^2 & 1 \\ 4 & (c^z-c^{-z})^2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & (a^x-a^{-x})^2 & 1 \\ 1 & (b^y-b^{-y})^2 & 1 \\ 1 & (c^z-c^{-z})^2 & 1 \end{vmatrix} = 0$$

Sol.66 A

Given, $\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

multiplying C_1, C_2, C_3 by a, b, c respectively

$$= \frac{1}{abc} \begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & ab^2 & ac^2 \\ a^2c & b^2c & c(c^2+a) \end{vmatrix}$$

Now taking common a, b, c from R_1, R_2, R_3 respectively

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} (c_1 \to c_1 + c_2 + c_3)$$

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} (R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

$= (1 + a^2 + b^2 + c^2)$ (1.1.1.)
$= (1 + a^2 + b^2 + c^2)$

Sol.67 D

Given a, b, c are non-zero real numbers then

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

Multiply R_1, R_2, R_3 by a, b, c respectively and hence divide by abc

$$= \frac{1}{abc} \begin{vmatrix} (abc)bc & abc & a(b+c) \\ (abc)ca & abc & b(c+a) \\ (abc)ab & abc & c(a+b) \end{vmatrix}$$

$$= \frac{(abc)^2}{abc} \begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ca+bc \end{vmatrix}$$

$$= abc \begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \qquad (c_3 \to c_1 + c_3)$$

$$= abc\,(ab + bc + ca) \begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} = 0$$

Sol.68 B

$$\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= 2 \begin{vmatrix} a_1+b_1+c_1 & c_1+a_1 & a_1+b_1 \\ a_2+b_2+c_2 & c_2+a_2 & a_2+b_2 \\ a_3+b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}$$

$= C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= 2 \begin{vmatrix} a_1+b_1+c_1 & -b_1 & -c_1 \\ a_2+b_2+c_2 & -b_2 & -c_2 \\ a_3+b_3+c_3 & -b_3 & -c_3 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= 2 \begin{vmatrix} a_1 & -b_1 & -c_1 \\ a_2 & -b_2 & -c_2 \\ a_3 & -b_3 & -c_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Sol.69 C

Given x, y, z $\in$ R

$$\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16$$

L.H.S. = $R_3 \to R_3 - R_1 - R_2$

$$= \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 0 & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} x & x+y & x+y+z \\ 0 & 3x & 5x+3y \\ 0 & x & x+y \end{vmatrix} \quad (R_2 \to R_2 - 2R_1)$$

$= x[3x(x + y) – x(5x + 3y)]$
$= – 2x^3$
Now $–2x^3 = – 16 \Rightarrow x = 2$

Sol.70 B

Given $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\sin\theta & \sin\theta & \cos\phi \end{vmatrix}$

Applying $R_1 \to R_1 + \sin\phi\, R_2 + \cos\phi\, R_3$

$$= \begin{vmatrix} \cos(\theta+\phi) + \sin\theta\sin\phi - \cos\theta\cos\phi & -\sin(\theta+\phi) + \cos\theta\sin\phi + \sin\theta\cos\phi & \cos 2\phi + \sin^2\phi + \cos^2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$$

$= (\cos 2\phi + 1)(\sin^2\theta + \cos^2\theta) = (1 + \cos 2\phi)$

Sol.71 A
Given

$$\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = k \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$(b^2 – a^2) (c^2 – a^2) (c^2 – b^2) = k (a – b) (b – c) (c – a)$
$k = (a + b) (b + c) (c + a)$

Sol.72 B
Given $a \neq b$
$ax + by + bz = 0$
$bx + ay + by = 0$
$bx + by + ax = 0$
for non trivial solution

$$D = 0 \Rightarrow \begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$(a + 2b) \begin{vmatrix} 1 & b & b \\ 1 & a & b \\ 1 & b & a \end{vmatrix} = 0$$

$R_2 \to R_2 – R_1$ & $R_3 \to R_3 – R_1$

$$\Rightarrow (a + 2b) \begin{vmatrix} 1 & b & b \\ 0 & a-n & 0 \\ 0 & 0 & a-b \end{vmatrix} = 0 \Rightarrow a + 2b = 0$$

Sol.73 A

$$\Delta = \begin{vmatrix} \sin 2\alpha & \sin(\alpha+\beta) & \sin(\alpha+1) \\ \sin(\beta+\alpha) & \sin 2\beta & \sin(\gamma+\beta) \\ \sin(\gamma+\alpha) & \sin(\gamma+\beta) & \sin 2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha\cos\alpha + \sin\alpha\cos\alpha & \sin\alpha\cos\beta\sin\beta & \sin\alpha\cos\gamma + \operatorname{cosec}\sin\gamma \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & \sin\beta\cos\beta + \sin\beta\cos\beta & \sin\gamma\cos\beta + \cos\sin\beta \\ \sin\gamma\cos\alpha + \cos\gamma\sin\alpha & \sin\gamma\cos\beta + \cos\beta & \sin\gamma\cos\gamma + \sin\gamma\cos\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} = 0$$

Sol.74 C

$$D = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$$

$$= abc \begin{vmatrix} a(1+x) & a & a \\ b & b(1+x) & b \\ c & c & c(1+x) \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+a) & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$;

$$= a^2b^2c^2 (3 + x) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix}$$

$R_2 \to R_2 \to R_1$; & $R_3 \to R_3 – R_1$;

$$= a^2b^2c^2 (x + 3) \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} \Rightarrow a^2b^2c^2 x^2 (x + 3)$$

Sol.75 B

$$D = \begin{vmatrix} \sin\frac{A}{2} & \sin\frac{B}{2} & \sin\frac{C}{2} \\ \sin(A+B+C) & \sin\left(\frac{B}{2}\right) & \sin\frac{A}{2} \\ \cos\left(\frac{A+B+C}{2}\right) & \tan(A+B+C) & \sin\frac{C}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\frac{A}{2} & \sin\frac{B}{2} & \sin\frac{C}{2} \\ \sin\pi & \sin\frac{B}{2} & \sin\frac{A}{2} \\ \cos\frac{\pi}{2} & \tan\pi & \sin\frac{C}{2} \end{vmatrix} = \begin{vmatrix} \sin\frac{A}{2} & \sin\frac{B}{2} & \sin\frac{C}{2} \\ 0 & \sin\frac{B}{2} & \sin\frac{A}{2} \\ 0 & \tan\pi & \sin\frac{C}{2} \end{vmatrix}$$

$$= D = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$D \le \frac{1}{8}$$

Sol.76 B

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$C_1 \to C_1 + C_2$

$$f(x) = \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 2 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$C_1 \to C_1 + C_3$

$$f(x) = (2 + 4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 1 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$f(x) = (2 + 4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$f(x) = 2 + 4\sin 2x$

$f'(x) = 8\cos 2x = 0 \Rightarrow \cos 2x = \cos\frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$

$$f\left(\frac{\pi}{4}\right) = \begin{vmatrix} \frac{3}{2} & \frac{1}{2} & 4 \\ \frac{1}{2} & \frac{3}{2} & 4 \\ \frac{1}{2} & \frac{1}{2} & 5 \end{vmatrix} = 6$$

Sol.77 A

$$D = \begin{vmatrix} a^3 - x & a^4 - x & a^5 - x \\ a^5 - x & a^6 - x & a^7 - x \\ a^7 - x & a^8 - x & a^9 - x \end{vmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 \to R_1$

$$D = \begin{vmatrix} a^3 - x & a^4 - x & a^5 - x \\ a^2 & a^2 & a^2 \\ a^4 & a^4 & a^4 \end{vmatrix}$$

$$D = a^2 \cdot a^4 \begin{vmatrix} a^3 - x & a^4 - x & a^5 - x \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$D = 0$

Sol.78 C

$$f(x) = \begin{vmatrix} a^{-x} & e^{x\ell na} & x^2 \\ a^{-3x} & e^{3x\ell na} & x^4 \\ a^{-5x} & e^{5x\ell na} & 1 \end{vmatrix}$$

$$f(x) = \begin{vmatrix} a^{-x} & a^{x} & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix}$$

$$f(-x) = \begin{vmatrix} a^{x} & a^{-x} & x^2 \\ a^{3x} & a^{-3x} & x^4 \\ a^{5x} & a^{-5x} & 1 \end{vmatrix}$$

$$f(x) = -\begin{vmatrix} a^{-x} & a^{x} & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix}$$

$f(x) + f(-x) = 0$

Sol.79 B

$$D = \begin{vmatrix} 1 & \frac{4\sin B}{b} & \cos A \\ 2a & 8\sin A & 1 \\ 3a & 12\sin A & \cos B \end{vmatrix}$$

From the property

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$$D = \begin{vmatrix} 1 & 4k & \cos A \\ 2a & 8ak & 1 \\ 3a & 12ak & \cos B \end{vmatrix}$$

$$D = 4k \begin{vmatrix} 1 & 1 & \cos A \\ 2a & 8a & 1 \\ 3a & 12a & \cos B \end{vmatrix}$$

$D = 4k \times 0 = 0$

Sol.80 C

$$\Delta_1 = \begin{vmatrix} 2a & b & e \\ 2d & e & t \\ 4x & 2y & 2z \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} f & 2d & e \\ 2z & 4x & 2y \\ e & 2a & b \end{vmatrix}$$

$$\Delta_1 = 4 \begin{vmatrix} a & b & e \\ d & e & f \\ x & y & z \end{vmatrix}, \ \Delta_2 = 4 \begin{vmatrix} f & d & e \\ z & x & y \\ e & a & b \end{vmatrix}$$

$$\Delta_1 = -4 \begin{vmatrix} Q & e & b \\ d & f & e \\ x & z & y \end{vmatrix} = 4 \begin{vmatrix} e & a & b \\ f & d & e \\ z & x & y \end{vmatrix}$$

$$\Delta_1 = -4 \begin{vmatrix} f & d & e \\ e & a & b \\ z & x & y \end{vmatrix} = 4 \begin{vmatrix} f & d & e \\ z & x & y \\ e & a & b \end{vmatrix} = \Delta_2$$

$\Delta_1 = \Delta_2 = 0$

Sol.81 B

$$\text{Given L.H.S.} = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & (a+b)^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Applying $c_2 - c_1$ and $c_3 - c_1$

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

Take out (a + b + c) common from each of

$$C_2 \text{ and } C_3 = (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $R_1 - (R_2 + R_3)$. Then

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $c_1 \to c_1 + \frac{1}{b} c_1$, $c_3 \to c_3 + \frac{1}{c} c_1$

$$= (a+b+c)^2 \begin{vmatrix} 2bc & 0 & 0 \\ b^2 & c+a & b^2/c \\ c^2 & c^2/b & a+b \end{vmatrix}$$

$= 2bc\,(a + b + c)^2\,[(a + c)\,(a + b) – bc]$

$= 2bc\,(a + b + c)^2\,(a^2 + ab + ac + bc – bc)2$

$= 2abc(a + b + c)^3$

Sol.82 B

$$\text{Given } U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix}$$

$$\sum_{n=1}^{n} U_n = \begin{vmatrix} \sum_{n=1}^{n} n & 1 & 5 \\ \sum_{n=1}^{N} n^2 & 2N+1 & 2N+1 \\ \sum_{n=1}^{N} n^3 & 3N^2 & 3N+1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{N(n+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left(\frac{N(N+1)}{2}\right)^2 & 3N^2 & 3N+1 \end{vmatrix}$$

$$= \frac{N(N+1)(2N+1)}{2} \begin{vmatrix} 1 & 1 & 5 \\ \frac{1}{3} & 1 & 1 \\ \frac{N(N+1)}{2} & 3N^2 & 3N+1 \end{vmatrix}$$

$$= \frac{N(N+1)(2N+1)}{3} \Rightarrow 2\sum_{n=1}^{N} n^2$$